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LETTER TO THE EDITOR

Envelope exact solutions for the generalized nonlinear Schrödinger equation with a source

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Abstract
In this letter, the generalized nonlinear Schrödinger (GNLS) equation, phase-locked a source $\kappa\,\mathrm{e}^{\mathrm{i}[\chi(\xi)-\omega t]}$: $\mathrm{i}\frac{\partial u}{\partial t} + a\frac{\partial^2 u}{\partial x^2} + bu|u|^2 + \mathrm{i}c\frac{\partial^3 u}{\partial x^3} + \mathrm{i}d\frac{\partial(u|u|^2)}{\partial x} = \kappa\,\mathrm{e}^{\mathrm{i}[\chi(\xi)-\omega t]}$, is investigated. Firstly, we reduce this equation to a second-order non-homogeneous nonlinear ordinary differential equation via a plane wave transformation and some constraint conditions. Then we use some fractional transformations to study exact solutions in obtaining the GNLS equation. As a consequence, many types of exact solutions are deduced such as envelope rational solutions, envelope periodic wave solutions, envelope solitary wave solutions and envelope doubly periodic solutions. Similarly, the corresponding exact solutions can also be obtained for the Hirota-type GNLS equation with a source and their combined equation.

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As is well known, the nonlinear Schrödinger (NLS) equation [1–4]

$$\mathrm{i}\frac{\partial u}{\partial t} + a\frac{\partial^2 u}{\partial x^2} + bu|u|^2 = 0 \tag{1}$$

plays an important role in many nonlinear sciences. It arises as an asymptotic limit of a slowly varying dispersive wave envelope in a nonlinear medium and as such has significant applications such as optical soliton communications, plasma physics, etc. Moreover, the NLS equation admits many remarkable properties, e.g., bright and dark soliton solutions, Lax pair, Liouville integrability, inverse scattering transformation, conservation laws, bilinearization, Painlevé integrability, multi-solitons, Bäcklund transformation, Darboux transformation, symmetries, etc. To improve the transmission rate in optical soliton communications, high-power and ultrashort optical pulses should be used. At the same time, higher-order dispersion

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and nonlinear effects should also be considered. Therefore, the following generalized nonlinear Schrödinger (GNLS) equation was presented [4, 6]:

$$\mathrm{i}\frac{\partial u}{\partial t}+a\frac{\partial^2 u}{\partial x^2}+bu|u|^2+\mathrm{i}c\frac{\partial^3 u}{\partial x^3}+\mathrm{i}d\frac{\partial(u|u|^2)}{\partial x}=0, \tag{2}$$

which contains a third-order dispersive term and a self-steepening term, where a, b, c, d are real constants. In addition, Hirota [5] introduced another GNLS equation in the form

$$\mathrm{i}\frac{\partial u}{\partial t}+a\frac{\partial^2 u}{\partial x^2}+bu|u|^2+\mathrm{i}c\frac{\partial^3 u}{\partial x^3}+\mathrm{i}h|u|^2\frac{\partial u}{\partial x}=0 \tag{3}$$

and presented its N-soliton solutions using the bilinear method, where a, b, c, h are real constants, and $ah = 3bc$. Zheng [6] had deduced an envelope solitary wave solution of (3) by gauge transformation and shown that (2) and (3) were different from the view of exact solutions.

Recently, Raju *et al* [7] studied the NLS equation with a source in the form

$$\mathrm{i}\frac{\partial \psi}{\partial t}+\frac{\partial^2 \psi}{\partial x^2}+g|\psi|^2\psi+\mu\psi=\kappa\,\mathrm{e}^{\mathrm{i}[\chi(\xi)-\omega t]}, \tag{4}$$

where g, μ, κ are real and $\xi = \alpha(x - vt)$. Moreover, some exact solitary wave solutions of (4) were given using a fractional transformation. Moreover, we can also obtain other types of solutions of (2)–(4) using some transformations and some of our methods [8].

In this letter, we will investigate the GNLS equation with a source in the form

$$\mathrm{i}\frac{\partial u}{\partial t}+a\frac{\partial^2 u}{\partial x^2}+bu|u|^2+\mathrm{i}c\frac{\partial^3 u}{\partial x^3}+\mathrm{i}d\frac{\partial(u|u|^2)}{\partial x}=\kappa\,\mathrm{e}^{\mathrm{i}[\chi(\xi)-\omega t]} \tag{5}$$

via fractional transformations and some ansatze, where $\xi = \alpha(x - vt)$; $\chi(\xi)$ is a real function and $a, b, c, d, \kappa, \alpha, v, \omega$ are all real.

To study envelope exact solutions of the GNLS equation with a source (5), we take a plane wave transformation in the form $u(x, t) = \phi(\xi)\,\mathrm{e}^{\mathrm{i}[\chi(\xi)-\omega t]}$, where $\phi(\xi)$ is a real function. For convenience, let $\chi(\xi) = \beta\xi + c_0$, where β, c_0 are real constants. Then, separating the real and imaginary parts of equation (5), respectively, we obtain the following two ordinary differential equations:

$$c\alpha^3\phi''' + (-\alpha v + 2a\alpha^2\beta - 3c\alpha^3\beta^2)\phi' + 3d\alpha\phi^2\phi' = 0, \tag{6}$$

$$(a\alpha^2 - 3c\alpha^3\beta)\phi'' + (\alpha\beta v + \omega - a\alpha^2\beta^2 + c\alpha^3\beta^3)\phi + (b - d\alpha\beta)\phi^3 - \kappa = 0. \tag{7}$$

Integrating equation (6) w.r.t. ξ once yields

$$c\alpha^2\phi'' + \sigma\phi + d\phi^3 - C = 0, \tag{8}$$

where $\sigma = -v + 2a\alpha\beta - 3c\alpha^2\beta^2$, and C is an integration constant.

Since the same function $\phi(\xi)$ satisfies two equations (7) and (8), we obtain the following constraint conditions:

$$\frac{a\alpha^2 - 3c\alpha^3\beta}{c\alpha} = \frac{\alpha\beta v + \omega - a\alpha^2\beta^2 + c\alpha^3\beta^3}{\sigma} = \frac{b - d\alpha\beta}{d} = \frac{\kappa}{C}. \tag{9}$$

In the following, with the aid of symbolic computation, we consider equation (8) via some transformations such that many corresponding types of exact solutions of the GNLS equation with a source (5) are obtained.

Case 1: rational wave solutions. We assume that (8) admits the solution $\phi(\xi) = \frac{A+B\xi^2}{E+F\xi^2}$, where A, B, E, F are constants to be determined later. The substitution of this expression into (8) can determine these parameters. Thus, we can obtain the envelope rational wave solution of (5):

$$u(x,t) = \frac{3c\alpha^2 F^3 + C\sqrt[3]{-2dC^2}\xi^2}{-c\alpha^2\sqrt[3]{-2dC^2} - 2dC^2\xi^2}\,\mathrm{e}^{\mathrm{i}[\beta\xi-\omega t+c_0]}, \tag{10}$$

where $\sigma = -3\sqrt[3]{\frac{1}{4}dC^2}$.

Case 2: periodic wave (trigonometric function) solutions. We assume that equation (8) admits the solution $\phi(\xi) = \frac{A+B\sin^2(\xi+\xi_0)}{E+F\sin^2(\xi+\xi_0)}$, where A, B, E, F are constants to be determined later. The substitution of this expression into equation (8) can determine these parameters. Therefore, we have the envelope periodic wave solution of equation (5):

$$u(x,t) = \frac{c\alpha(3F+2) - c\alpha F(F+2)\sin^2(\xi+\xi_0)}{\sqrt{\frac{2c}{d(1+F)}}[1 + F\sin^2(\xi+\xi_0)]}\,\mathrm{e}^{\mathrm{i}[\beta\xi-\omega t+c_0]}, \tag{11}$$

where

$$\sigma = -\frac{c\alpha^2(3F^2+4F+4)}{2(1+F)}, \qquad C = \frac{c^2\alpha^3F^2(F+2)}{2d(F+1)^2}\sqrt{\frac{2d(F+1)}{c}}.$$

In particular, we have the periodic wave solution of (5):

$$u(x,t) = -4\alpha\sqrt{\frac{2c}{3d}}\frac{\sin^2(\xi+\xi_0)}{3 - 2\sin^2(\xi+\xi_0)}\,\mathrm{e}^{\mathrm{i}[\beta\xi-\omega t+c_0]}, \tag{12}$$

where

$$\sigma = -4c\alpha^2, \qquad C = -\frac{8}{3}c\alpha^3\sqrt{\frac{2c}{3d}}.$$

Case 3: solitary wave (hyperbolic function) solutions. We assume that (8) admits the solution $\phi(\xi) = \frac{A+B\cosh^2(\xi)}{E+F\cosh^2(\xi)}$, where A, B, E, F are constants to be determined later. The substitution of this expression into (8) can determine these parameters. Thus, we obtain the envelope solitary wave solution of (5):

$$u(x,t) = \frac{-c\alpha(3F+2)\,\mathrm{sech}^2(\xi) + c\alpha F(F+2)}{d(F+1)\sqrt{-\frac{2c}{d(1+F)}}[\mathrm{sech}^2(\xi) + F]}\,\mathrm{e}^{\mathrm{i}[\beta\xi-\omega t+c_0]}, \tag{13}$$

where

$$\sigma = \frac{c\alpha^2(3F^2+4F+4)}{2(1+F)}, \qquad C = \frac{c^2\alpha^3F^2(F+2)}{2d(F+1)^2}\sqrt{-\frac{2d(F+1)}{c}}.$$

In particular, we have the dark solitary wave solution of (5):

$$u(x,t) = 4\alpha\sqrt{-\frac{2c}{3d}}\frac{\mathrm{e}^{\mathrm{i}[\beta\xi-\omega t+c_0]}}{3\,\mathrm{sech}^2(\xi) - 2}, \tag{14}$$

where

$$\sigma = 4c\alpha^2, \qquad C = -\frac{8}{3}c\alpha^3\sqrt{-\frac{2c}{3d}}.$$

Case 4: solitary wave (hyperbolic function) solutions. We assume that (8) admits the solution $\phi(\xi) = \frac{A+B\sinh^2(\xi)}{E+F\sinh^2(\xi)}$, where A, B, E, F are constants to be determined later. The substitution of this expression into (8) can determine these parameters. Therefore, we get the envelope solitary wave solution of (5)

$$u(x,t) = \frac{c\alpha(3F-2) - c\alpha F(F-2)\sinh^2(\xi)}{d(F-1)\sqrt{\frac{2c}{d(F-1)}}[1+F\sinh^2(\xi)]}\,\mathrm{e}^{\mathrm{i}[\beta\xi-\omega t+c_0]}, \tag{15}$$

where

$$\sigma = -\frac{c\alpha^2(3F^2-4F+4)}{2(F-1)}, \qquad C = \frac{c^2\alpha^3F^2(F-2)}{2d(F-1)^2}\sqrt{\frac{2d(F-1)}{c}}.$$

In particular, we have a singular envelope solitary wave solution:

$$u(x,t) = 4\alpha\sqrt{-\frac{2c}{3d}}\frac{\mathrm{e}^{\mathrm{i}[\beta\xi-\omega t+c_0]}}{3\,\mathrm{csch}^2(\xi)+2}, \tag{16}$$

where

$$\sigma = 4c\alpha^2, \qquad C = \frac{8}{3}c\alpha^3\sqrt{-\frac{2c}{3d}}.$$

Case 5: doubly periodic wave (Jacobian elliptic function) solutions. We assume that (8) admits the solution $\phi(\xi) = \frac{A+B\mathrm{cn}^2(\xi;m)}{E+F\mathrm{cn}^2(\xi;m)}$, where A, B, E, F are constants to be determined later. The substitution of this expression into (8) can determine these parameters. Therefore, we obtain the envelope doubly periodic wave solutions of (5):

$$u(x,t) = 2\alpha\sqrt{\frac{4c(m^4-m^2+1)F+2cm^2(2m^2-1)}{9d(m^2-1)}}\frac{\mathrm{cn}^2(\xi;m)}{1+F\mathrm{cn}^2(\xi;m)}\,\mathrm{e}^{\mathrm{i}[\beta\xi-\omega t+c_0]}, \tag{17}$$

where

$$F = \frac{(1-2m^2)\pm\sqrt{4-32m+76m^2-12m^4}}{3(m^2-1)},$$
$$C = -4c\alpha^2\sqrt{\frac{4c(m^4-m^2+1)F+2cm^2(2m^2-1)}{9d(m^2-1)}},$$
$$\sigma = -4c\alpha^2(3m^2F-3F+2m^2-1).$$

Another envelope doubly periodic wave solutions of (5) is

$$u(x,t) = 2\alpha\sqrt{\frac{4c(1-m^2)F+2c(1-2m^2)}{3d}}\frac{1}{1+F\mathrm{cn}^2(\xi;m)}\,\mathrm{e}^{\mathrm{i}[\beta\xi-\omega t+c_0]}, \tag{18}$$

where

$$F = \frac{(1-2m^2)\pm\sqrt{4-32m+76m^2-12m^4}}{m^2-1},$$
$$C = \frac{4}{3}c\alpha^3\sqrt{\frac{4c(1-m^2)F+2c(1-2m^2)}{3d}}(m^2F-F+4m^2-2),$$
$$\sigma = 4c\alpha^2(m^2F-F+2m^2-1).$$

Similarly, we can also consider the Hirota-type GNLS equation with a source in the form

$$\mathrm{i}\frac{\partial u}{\partial t} + a\frac{\partial^2 u}{\partial x^2} + bu|u|^2 + \mathrm{i}c\frac{\partial^3 u}{\partial x^3} + \mathrm{i}h|u|^2\frac{\partial u}{\partial x} = \kappa\,\mathrm{e}^{\mathrm{i}[\chi(\xi)-\omega t]}. \tag{19}$$

We take the transformation in the form

$$u(x,t) = \phi(\xi)\,\mathrm{e}^{\mathrm{i}[\chi(\xi)-\omega t]}, \qquad \chi(\xi) = \beta\xi + c_0. \tag{20}$$

Then, separating the real and imaginary parts of (19), respectively, we have the following two ordinary differential equations:

$$c\alpha^3\phi''' + (-\alpha v + 2a\alpha^2\beta - 3c\alpha^3\beta^2)\phi' + h\alpha\phi^2\phi' = 0, \tag{21}$$

$$(a\alpha^2 - 3c\alpha^3\beta)\phi'' + (\alpha\beta v + \omega - a\alpha^2\beta^2 + c\alpha^3\beta^3)\phi + (b - h\alpha\beta)\phi^3 - \kappa = 0, \tag{22}$$

which are similar to equations (6) and (7). Therefore, we can also obtain the corresponding types of envelope exact solutions of the Hirota-type GNLS equation with a source (19). We omit them here.

In addition, the combined equation of the GNLS equation with a source (5) and the Hirota-type GNLS equation with a source is described by

$$\mathrm{i}\frac{\partial u}{\partial t} + a\frac{\partial^2 u}{\partial x^2} + bu|u|^2 + \mathrm{i}c\frac{\partial^3 u}{\partial x^3} + \mathrm{i}d\frac{\partial(u|u|^2)}{\partial x} + \mathrm{i}h|u|^2\frac{\partial u}{\partial x} = \kappa\,\mathrm{e}^{\mathrm{i}[\chi(\xi)-\omega t]}. \tag{23}$$

It is easy to know that (i) when $\kappa = 0, h = 0$, equation (23) reduces to the GNLS equation (2); (ii) when $\kappa = 0, d = 0$, equation (23) becomes the Hirota-type GNLS equation (3); when $h = 0$, equation (23) reduces to the GNLS equation with a source (5) and (iv) when $d = 0$, equation (23) reduces to the Hirota-type GNLS equation with a source (19). Similarly, if we take the transformation (20), then separating the real and imaginary parts of (23) yields the following two ordinary differential equations, respectively:

$$c\alpha^3\phi''' + (-\alpha v + 2a\alpha^2\beta - 3c\alpha^3\beta^2)\phi' + (3d + h)\alpha\phi^2\phi' = 0, \tag{24}$$

$$(a\alpha^2 - 3c\alpha^3\beta)\phi'' + (\alpha\beta v + \omega - a\alpha^2\beta^2 + c\alpha^3\beta^3)\phi + (b - d\alpha\beta - h\alpha\beta)\phi^3 - \kappa = 0. \tag{25}$$

It is obvious to see that (24) and (25) have the same structures as (6) and (7) except for the coefficients. Therefore, we can also deduce the corresponding envelope exact solutions of (23), which are omitted here.

In summary, we have obtained some new envelope exact solutions of the GNLS equation with a source (5) using a plane wave transformation and some fractional transformations. These solutions may be useful in further understanding the GNLS equation with a source. Similarly, the corresponding solutions of the Hirota-type GNLS equation with a source (19) and their combined equation (23) can be obtained. These transformation may be used to study the solutions of other nonlinear wave equations.

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References

[1] Kaup D J and Newell A C 1978 *Proc. R. Soc.* A **361** 413

[2] Abloiwtz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)
Abloiwtz M J and Segur H 1981 *Solitons and Inverse Scattering Transformation* (Philadelphia: SIAM)

[3] Drazin P G and Johnson R S 1989 *Solitons: An Introduction* (Cambridge: Cambridge University Press)

[4] Agrawal P G 1989 *Nonlinear Fiber Optics* (New York: Academic)
[5] Hirota R 1978 *J. Math. Phys.* **14** 805
[6] Zheng W 1994 *J. Phys. A: Math. Gen.* **27** L931
[7] Raju T S *et al* 2005 *J. Phys. A: Math. Gen.* **38** L271
[8] Yan Z 2005 *Chaos Solitons Fractals* **23** 767
Yan Z 2001 *Phys. Lett.* A **292** 100